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On a generalization of a theorem of Erdős and Fuchs

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ABSTRACT

Let $A = \{a_1, a_2, \ldots\}(a_1 < a_2 < \cdots)$ be an infinite sequence of nonnegative integers, let $k \geq 2$ be a fixed integer and denote by $r_k(A, n)$ the number of solutions of $a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq n$. Montgomery and Vaughan proved that $r_2(A, n) = cn + o(n^{1/4})$ cannot hold for any constant $c > 0$. In this paper, we extend this result to $k > 2$.

1. Introduction

Let $k \geqslant 2$ be a fixed integer and let $A = \{a_1, a_2, \ldots\}(a_1 < a_2 < \cdots)$ be an infinite sequence of nonnegative integers. We write $F(z) = \sum_{a \in A} z^a$, $A(n) = \sum_{a \in A, a \leqslant n} 1$. For $n = 0, 1, 2 \cdots$ let $r_k(A, n)$ denote the number of solutions of

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leqslant n, \quad a_{i_1} \in A, a_{i_2} \in A, \ldots, a_{i_k} \in A.$$

In 1956, Erdős and Fuchs [1] proved the following result:

Theorem A. *If $A \subset \mathbb{N}$, then*

$$r_2(A, n) = cn + o(n^{1/4}(\log n)^{-1/2})$$

cannot hold for any constant $c > 0$.

Jurkat (unpublished), and later Montgomery and Vaughan [5] improved the Erdős–Fuchs theorem by eliminating the log power on the right-hand side:

Theorem B. *If $A \subset \mathbb{N}$, then*

$$r_2(A, n) = cn + o(n^{1/4})$$

cannot hold for any constant $c > 0$.

Already, the Erdős–Fuchs theorem has been extended in various directions. In [6], Sárkőzy generalized this theorem for two arbitrary sequences which are "near" in a certain sense; he proved the following theorem:

Theorem C. *Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be infinite sequences of integers such that $0 \leqslant a_1 < a_2 < \cdots$ and $0 \leqslant b_1 < b_2 < \cdots$. If*

$$a_i - b_i = o\left(\frac{{a_i}^{1/2}}{\log a_i}\right),$$

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then

$$|\{(i,j) : a_i + b_j \leqslant n\}| = cn + o(n^{1/4}(\log n)^{-1/2})$$

cannot hold for any constant $c > 0$.

In 2004, using the idea of Jurkat (differentiation of the generating function), Horváth [2] extended similarly the result of Montgomery and Vaughan for "sufficiently near" sequences:

Theorem D. *If* $A = \{a_1, a_2, \ldots\}$ *and* $B = \{b_1, b_2, \ldots\}$ *(where* $0 \leqslant a_1 < a_2 < \cdots$, $0 \leqslant b_1 < b_2 < \cdots$*) are infinite sequences of integers such that*

$$a_i - b_i = o(a_i^{1/2})$$

and

$$A(n) - B(n) = O(1),$$

then

$$|\{(i,j) : a_i + b_j \leqslant n\}| = cn + o(n^{1/4})$$

cannot hold for any constant $c > 0$.

In 2002, Horváth [4] extended the Erdős–Fuchs theorem further by considering sums $A^{(1)} + A^{(2)} + \cdots + A^{(k)}$. In particular, for the case $A^{(1)} \equiv A^{(2)} \equiv \cdots \equiv A^{(k)}$, Horváth's result implies that if $A \subset \mathbb{N}$, then

$$r_k(A,n) = cn + o\left(n^{1/4}(\log n)^{1-3k/4}\right)$$

cannot hold for any constant $c > 0$.

In this paper, we obtain the following result:

Theorem. *If* $A \subset \mathbb{N}$ *and* $k > 2$, *then*

$$r_k(A,n) = cn + o(n^{1/4}) \tag{1}$$

cannot hold for any constant $c > 0$.

Throughout this paper, let $z = re(\alpha)$, where $e(\alpha) = \mathrm{e}^{2\pi \mathrm{i}\alpha}$ and $r = 1 - \frac{1}{N}$. N is a large positive integer; α is a real number.

2. Lemmas

Lemma 1 (*[3]*). *let* $2 < m = m(N) < N$, *where* m *is a positive integer, and* $m \to \infty$ *as* $N \to \infty$. *Then*

$$\int_0^1 |1-z|^{-\beta} \left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha \ll \begin{cases} m^{\beta+1} & \text{if } 0 \leqslant \beta < 1, \\ m^2 \log N & \text{if } \beta = 1, \\ m^2 N^{\beta-1} & \text{if } 1 < \beta. \end{cases}$$

Lemma 2 (*[4]*). *Let* $r = 1 - \frac{1}{N}$, *where* N *is a large positive integer. Then:*

(a) $\sum_{n=0}^{\infty} nr^{2n} \leqslant N^2$.
(b) $\sum_{n=0}^{\infty} (n+1)^4 r^{2n} \ll N^5$.

3. Proof of theorem

Suppose that (1) holds. Let $\vartheta(n) = r_k(A,n) - cn$; then for $|z| < 1$, we have

$$\frac{1}{1-z}F^k(z) = \sum_{n=0}^{\infty} r_k(A,n)z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty} \vartheta(n)z^n,$$

$$F^k(z) = \frac{cz}{1-z} + (1-z)\sum_{n=0}^{\infty} \vartheta(n)z^n. \tag{2}$$

Using the idea of Jurkat, by differentiation of (2),

$$kF^{k-1}(z)F'(z) = \frac{c}{(1-z)^2} - \sum_{n=0}^{\infty} \vartheta(n)z^n + (1-z)\sum_{n=0}^{\infty} (n+1)\vartheta(n+1)z^n. \tag{3}$$

Letting ε be a fixed small positive number, $m=[\varepsilon N^{1/2}]$, and letting

$$J=\int_0^1 \left|kF^{k-1}(z)F'(z)\right|\cdot\left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha,$$

$$J_1=c\int_0^1 \frac{1}{|1-z|^2}\cdot\left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha,$$

$$J_2=\int_0^1 \left|\sum_{n=0}^{\infty}\vartheta(n)z^n\right|\cdot\left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha,$$

$$J_3=\int_0^1 \left|(1-z)\sum_{n=0}^{\infty}(n+1)\vartheta(n+1)z^n\right|\cdot\left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha,$$

by (3), we have

$$J\leqslant J_1+J_2+J_3. \tag{4}$$

Note that

$$\left|\frac{1-z^m}{1-z}\right|^2=\left|\sum_{t=0}^{m-1}z^t\right|^2=\sum_{t=0}^{m-1}r^te(t\alpha)\cdot\sum_{t=0}^{m-1}r^te(-t\alpha),$$

and $k>2$, so

$$\begin{aligned}J &> \int_0^1\left|\frac{1}{z}\sum_{a\in A}az^a\cdot\overline{\sum_{a\in A}z^a}\right|\cdot|F^{k-2}(z)|\cdot\left|\frac{1-z^m}{1-z}\right|^2\mathrm{d}\alpha\\ &> \frac{1}{r}\left|\int_0^1\left(\sum_{a\in A}ar^ae(a\alpha)\cdot\sum_{a\in A}r^ae(-a\alpha)\cdot\sum_{t=0}^{m-1}r^te(-t\alpha)\right)\cdot\left(F^{k-2}(z)\cdot\sum_{t=0}^{m-1}r^te(t\alpha)\right)\mathrm{d}\alpha\right|.\end{aligned}$$

Let

$$\sum_{\mu=-\infty}^{\infty}S_\mu e(\mu\alpha)=\sum_{a\in A}ar^ae(a\alpha)\cdot\sum_{a\in A}r^ae(-a\alpha)\cdot\sum_{t=0}^{m-1}r^te(-t\alpha),$$

$$\sum_{\gamma=0}^{\infty}T_\gamma e(\gamma\alpha)=F^{k-2}(z)\sum_{t=0}^{m-1}r^te(t\alpha).$$

It is obvious that all the coefficients S_μ, T_γ are nonnegative, so

$$\begin{aligned}J &> \left|\int_0^1\sum_{\mu=-\infty}^{\infty}S_\mu e(\mu\alpha)\cdot\sum_{\gamma=0}^{\infty}T_\gamma e(\gamma\alpha)\mathrm{d}\alpha\right|\\ &= \sum_{\mu+\gamma=0}S_\mu T_\gamma\\ &\geqslant \sum_{\frac{m}{4}\leqslant\gamma\leqslant m/2}S_{-\gamma}T_\gamma.\end{aligned} \tag{5}$$

If $\frac{m}{4}\le\gamma\le\frac{m}{2}<N$, noting that $r^N=(1-\frac{1}{N})^N\to\frac{1}{e}$, we have

$$\begin{aligned}T_\gamma &= \sum_{\substack{a_{i_3}+\cdots+a_{i_k}+t=\gamma\\ 0\le t\le m-1}} r^{a_{i_3}+\cdots+a_{i_k}+t}\ge r^N\sum_{\substack{a_{i_3}+\cdots+a_{i_k}+t=\gamma\\ 0\le t\le m/2}}1\\ &\gg \sum_{a_{i_3}+\cdots+a_{i_k}\le m/4}1\ge\left(\sum_{a_{i_j}\le\frac{m}{4(k-2)}}1\right)^{k-2}.\end{aligned}$$

and

$$\left(\sum_{a_{i_j}\le\frac{m}{4(k-2)}}1\right)^k=\left(A\left(\frac{m}{4(k-2)}\right)\right)^k\geqslant\sum_{a_{i_1}+\cdots+a_{i_k}\leqslant\frac{m}{4(k-2)}}1=r_k\left(A,\frac{m}{4(k-2)}\right).$$

By the indirect assumption, $r_k\left(A, \frac{m}{4(k-2)}\right) \sim cm$; thus if $\frac{m}{4} \leqslant \gamma \leqslant \frac{m}{2}$, then

$$T_\gamma \gg m^{\frac{k-2}{k}}. \tag{6}$$

Note that

$$r_k(A, N) = \sum_{\substack{a_{i_1}+\cdots+a_{i_k}\le N\\ a_{i_1},\ldots,a_{i_k}\in A}} 1 \le A^k(N) \le \sum_{\substack{a_{i_1}+\cdots+a_{i_k}\le kN\\ a_{i_1},\ldots,a_{i_k}\in A}} = r_k(A, kN),$$

and thus, by the indirect assumption, there exist positive numbers c_1 and c_2 such that for all sufficiently large N, $c_1N^{1/k} \le A(N) \le c_2N^{1/k}$. And

$$\begin{aligned}\sum_{\mu=-\infty}^{\infty} S_\mu e(\mu\alpha) &= \sum_{a_i\in A}\sum_{a_j\in A}\sum_{t=0}^{m-1} a_i r^{a_i+a_j+t} e((a_i-a_j-t)\alpha)\\ &= \sum_{\substack{a_i-a_j-t=\mu\\ 0\le t\le m-1,a_i,a_j\in A}} a_i r^{a_i+a_j+t} e(\mu\alpha).\end{aligned}$$

Therefore, if $\frac{m}{4} \leqslant \gamma \leqslant \frac{m}{2}$, we have

$$\begin{aligned}S_{-\gamma} &= \sum_{\substack{a_i-a_j-t=-\gamma\\ 0\le t\le m-1,a_i,a_j\in A}} a_i r^{a_i+a_j+t} \ge \sum_{\substack{a_i\in A\\ \left[\frac{c_1^kN}{2^kc_2^k}\right]<a_i\le N}} a_i r^{2a_i+\gamma}\\ &> r^{3N} \sum_{\substack{a_i\in A\\ \left[\frac{c_1^kN}{2^kc_2^k}\right]<a_i\le N}} a_i \gg N \sum_{\substack{a_i\in A\\ \left[\frac{c_1^kN}{2^kc_2^k}\right]<a_i\le N}} 1\\ &= N\cdot\left(A(N) - A\left(\left[\frac{c_1^kN}{2^kc_2^k}\right]\right)\right)\\ &\ge N\cdot\left(c_1N^{1/k} - c_2\left(\frac{c_1^kN}{2^kc_2^k}\right)^{1/k}\right) = \frac{1}{2}c_1N^{1+1/k} \gg N^{1+1/k}.\end{aligned} \tag{7}$$

By (5)–(7), we have

$$J \gg m\cdot m^{\frac{k-2}{k}}\cdot N^{1+\frac{1}{k}} = m^{2-\frac{2}{k}}\cdot N^{1+\frac{1}{k}}. \tag{8}$$

Now we estimate J_1, J_2, J_3.
By Lemma 1,

$$J_1 \ll m^2N. \tag{9}$$

By Cauchy's inequality and Parseval's formula,

$$\begin{aligned}J_2 &\ll m^2\int_0^1\left|\sum_{n=0}^{\infty}\vartheta(n)z^n\right|\mathrm{d}\alpha\\ &\leqslant m^2\left(\int_0^1\left|\sum_{n=0}^{\infty}\vartheta(n)z^n\right|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\\ &= m^2\left(\sum_{n=0}^{\infty}|\vartheta(n)|^2r^{2n}\right)^{\frac{1}{2}}.\end{aligned}$$

By the indirect assumption, and Lemma 2(a), we have

$$J_2 \ll m^2\left(\sum_{n=0}^{\infty}nr^{2n}\right)^{1/2} \leqslant m^2(N^2)^{\frac{1}{2}} = m^2N. \tag{10}$$

Similarly,

$$\begin{aligned} J_3 &\ll \int_0^1 \left|\sum_{n=0}^{\infty}(n+1)\vartheta(n+1)z^n\right| \cdot \left|\frac{1-z^m}{1-z}\right| \mathrm{d}\alpha \\ &\leqslant \left(\int_0^1 \left|\sum_{n=0}^{\infty}(n+1)\vartheta(n+1)z^n\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \left|\frac{1-z^m}{1-z}\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{\infty}(n+1)^2\vartheta^2(n+1)r^{2n}\right)^{\frac{1}{2}} \cdot \left(\sum_{j=0}^{m-1} r^{2j}\right)^{\frac{1}{2}} \\ &\leqslant m^{1/2}\left(\sum_{n=0}^{\infty}(n+1)^2\vartheta^2(n+1)r^{2n}\right)^{\frac{1}{2}}. \end{aligned} \tag{11}$$

Furthermore, by Cauchy's inequality and Lemma 2(b),

$$\begin{aligned} \sum_{n=0}^{\infty}(n+1)^2\vartheta^2(n+1)r^{2n} &\leqslant \left(\sum_{n=0}^{\infty}(n+1)^4 r^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty}\vartheta^4(n+1)r^{2n}\right)^{1/2} \\ &\ll N^{5/2}\left(\sum_{n=0}^{\infty}\vartheta^4(n+1)r^{2n}\right)^{1/2}. \end{aligned} \tag{12}$$

By the indirect assumption, for every $\varepsilon > 0$, there exists a natural number k such that for all $n \geqslant k$, $|\vartheta(n)| \leqslant \varepsilon n^{1/4}$. Then for $N > N(\varepsilon)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\vartheta^4(n+1)r^{2n} &\leqslant \sum_{n=0}^{k-1}\vartheta^4(n+1) + \sum_{n=1}^{\infty}\varepsilon^4(n+1)r^{2n} \\ &\leqslant \sum_{n=0}^{k-1}\vartheta^4(n+1) + 2\varepsilon^4\sum_{n=0}^{\infty} nr^{2n} \\ &\ll \sum_{n=0}^{k-1}\vartheta^4(n+1) + 2\varepsilon^4 N^2 \\ &\leqslant 3\varepsilon^4 N^2, \end{aligned}$$

and thus

$$\sum_{n=0}^{\infty}\vartheta^4(n+1)r^{2n} = o(N^2). \tag{13}$$

By (11)–(13), we have

$$J_3 \ll m^{1/2}\left(N^{5/2}(o(N^2))^{\frac{1}{2}}\right)^{\frac{1}{2}} = o(m^{1/2}N^{\frac{7}{4}}). \tag{14}$$

By (4), (8)–(10), (14),

$$m^{2-\frac{2}{k}}N^{1+\frac{1}{k}} \ll m^2 N + o(m^{1/2}N^{\frac{7}{4}}). \tag{15}$$

Since $m = [\varepsilon N^{1/2}]$, (15) yields

$$\varepsilon N^2 \ll \varepsilon^2 N^2 + o(\varepsilon^{1/2}N^2)$$

for all sufficiently large N; hence $\varepsilon \ll \varepsilon^2 + o(\varepsilon^{1/2})$. Thus $1 \ll \varepsilon$; but this cannot hold for sufficiently small ε.

This completes the proof of the theorem. □

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